Research Report

The Art of Counterpoint: Mathematical Three-voice First-species Counterpoint Model

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Abstract

In this paper we extend Mazzola's two-voice counterpoint theory and Hichert's admitted-successor construction to three-voice first-species counterpoint.

1. Introduction

It is sometimes said that the music of J. S. Bach is mathematical. One reason for this characterization lies in the compositional art of counterpoint, which Bach refined throughout his life. However, the history of describing counterpoint in a genuinely mathematical sense is not particularly long.

A systematic mathematical theory of counterpoint was established only with the emergence of Mazzola's model of counterpoint and the algorithm due to Hichert. Mazzola's mathematical theory of counterpoint provides an algebraic model of first-species counterpoint in terms of affine symmetries of dual-number extensions of pitch-class rings. Agustín-Aquino and collaborators have substantially extended this framework. Their work generalizes the original twelve-tone setting to microtonal and $2k$-tone equal-tempered systems, develops the algebraic and combinatorial theory of strong dichotomies, quasipolarities, and counterpoint symmetries, and organizes the resulting structures into the notion of counterpoint worlds. Further extensions include continuous counterpoint on the octave continuum, categorical and topos-theoretic generalizations of the underlying dichotomy-based formalism, and projection-oriented models for second-species counterpoint. These developments have also been used in compositional and analytical contexts. Thus, Agustín-Aquino's contributions transform Mazzola's original model from a theory of Fuxian first-species counterpoint into a general algebraic framework for constructing and comparing contrapuntal systems.

We propose a three-voice extension of Mazzola's first-species counterpoint model by retaining the ordinary two-voice counterpoint structure on the two bass-rooted projections and supplementing it with a harmonic relation between them. A three-voice sonority is represented by a lower voice together with two intervals, one from the lower voice to the middle voice and one from the lower voice to the upper voice. The admissible vertical sonorities are determined by requiring both bass-rooted intervals to be consonant and by imposing an additional harmonic relation on their ordered pair. This construction is a fibered three-voice extension of the ordinary two-voice Mazzola system over a common lower voice, rather than a new strong dichotomy on the full three-voice space. The harmonic relation selects the admissible pairs of bass-rooted consonances, allowing chordal configurations such as incomplete and first-inversion sonorities that are not captured by imposing the original dichotomy on every pair of voices. The successor relation extends Hichert's admitted-successor algorithm: the two bass-rooted projections follow the ordinary two-voice successor relation, and the target sonority must satisfy the harmonic relation. In the symmetric version, the middle–upper projection is also governed by the same two-voice mechanism whenever it is consonant, so inherited pairwise prohibitions, such as the Fuxian avoidance of proper parallel fifths, require no separate ad hoc rule. In the twelve-tone Fuxian case, the model uses the Renaissance consonances and a harmonic relation that admits complete and incomplete three-voice sonorities while excluding pairs whose middle–upper separation is a second. This yields a directed successor structure on the admissible three-voice Fuxian sonorities by maximizing compatible deformations in the spirit of Hichert's algorithm.

2. Three-Voice First-Species Counterpoint

Let $\mathbb{Z}_{2k}$ be the cyclic group of pitch classes in a $2k$-tone equal temperament, and let $\varepsilon.\mathbb{Z}_{2k}$ denote the corresponding module of interval classes. A marked interval dichotomy is a partition

$$(X/Y), \quad Y = X^c, \quad |X| = |Y| = k,$$

where X is interpreted as the set of consonances and Y as the set of dissonances. The affine group

$$\overrightarrow{\mathrm{GL}}(\mathbb{Z}_{2k}) = \{T_u \cdot v \mid u \in \mathbb{Z}_{2k},\ v \in \mathbb{Z}_{2k}^{\times}\},$$

$$(T_u \cdot v)(z) = u + vz,$$

acts on interval dichotomies. A dichotomy is called strong if it is both autocomplementary and rigid. Equivalently, for the present purpose, a strong dichotomy has a unique affine symmetry $p = T_u \cdot v$ such that $p(X) = Y$. This symmetry is called the polarity of the dichotomy.

Definition 2.1. Let

$$\mathbb{Z}_{2k}[\varepsilon_1, \varepsilon_2] = \{a+\varepsilon_1.b+\varepsilon_2.c : a, b, c \in \mathbb{Z}_{2k}\}, \quad \varepsilon_i\varepsilon_j = 0$$

for $i, j \in \{1, 2\}$. We write

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} := a + \varepsilon_1.b + \varepsilon_2.c.$$

The entries b and c are respectively the lower–middle and lower–upper intervals; the middle–upper interval is $c - b$.

Definition 2.2. Let

$$H := \begin{bmatrix} c \\ b \end{bmatrix} \subseteq X \times X$$

be fixed. We call H the three-voice harmonic mask associated with X. Define

$$X_H = \left\{ \begin{bmatrix} c \\ b \\ a \end{bmatrix} : a \in \mathbb{Z}_{2k},\ \begin{bmatrix} c \\ b \end{bmatrix} \in H \right\} \subseteq \mathbb{Z}_{2k}[\varepsilon_1, \varepsilon_2].$$

Thus a three-voice consonance is a pair of X-consonant intervals above a common lower voice, subject to the additional mask H.

2.1. Three-voice admitted successors

Remark 2.3 (Review of Two-voice admitted successors). Let

$$\Delta = (X/Y)$$

be a strong interval dichotomy of $\mathbb{Z}_{2k}$, with polarity

$$p = T_r \cdot w.$$

The induced two-voice counterpoint dichotomy is

$$X[\varepsilon] = \mathbb{Z}_{2k} + \varepsilon.X, \quad Y[\varepsilon] = \mathbb{Z}_{2k} + \varepsilon.Y.$$

This is the dual-number construction of the first-species model in Mazzola's theory. The corresponding induced polarity over the fibre $x + \varepsilon.\mathbb{Z}_{2k}$ is

$$p_x[\varepsilon] = T_{x(1-w)+\varepsilon.r} \cdot w.$$

Let

$$\xi = x + \varepsilon.k \in X[\varepsilon].$$

A symmetry $g \in \overrightarrow{GL}(\mathbb{Z}_{2k}[\varepsilon])$ is called a contrapuntal symmetry for ξ if the following conditions hold:

$$\xi \in gY[\varepsilon],$$

$$p_x[\varepsilon]\big(gX[\varepsilon]\big) = gY[\varepsilon],$$

and the cardinality

$$|gX[\varepsilon] \cap X[\varepsilon]|$$

is maximal among all symmetries satisfying the preceding two conditions.

Denote by

$$\Sigma_X(\xi)$$

the set of contrapuntal symmetries for ξ. The two-voice Mazzola–Hichert admitted-successor set of ξ is

$$S_X(\xi) = \bigcup_{g \in \Sigma_X(\xi)} \big(gX[\varepsilon] \cap X[\varepsilon]\big).$$

Definition 2.4. Define the three pairwise projections

$$\pi_{LM} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = a + \varepsilon.b, \quad \pi_{LU} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = a + \varepsilon.c,$$

and

$$\pi_{MU} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = (a + b) + \varepsilon.(c - b).$$

Definition 2.5. For

$$\xi = \begin{bmatrix} c \\ b \\ a \end{bmatrix} \in X_H,$$

define its active set by

$$A(\xi) = \{\alpha \in \{LM, LU, MU\} : \pi_\alpha(\xi) \in X[\varepsilon]\}.$$

Since $H \subseteq X \times X$, one has $LM, LU \in A(\xi)$. Moreover,

$$MU \in A(\xi) \iff c - b \in X.$$

Definition 2.6. The symmetric three-voice admitted-successor set of ξ is

$$S_H(\xi) = \{\eta \in X_H : \pi_\alpha(\eta) \in S_X(\pi_\alpha(\xi)) \quad \forall \alpha \in A(\xi)\}.$$

A symmetric three-voice first-species counterpoint over $(\mathbb{Z}_{2k}, X, H)$ is a finite sequence

$$\xi_1, \xi_2, \ldots, \xi_n$$

such that

$$\xi_i \in X_H$$

for every i, and

$$\xi_{i+1} \in S_H(\xi_i)$$

for every $1 \le i < n$.

Proposition 2.7. *Let $q \in X$. Suppose that the two-voice admitted-successor correspondence S_X forbids non-trivial parallel q-motion, i.e.*

$$x' + \varepsilon.q \notin S_X(x + \varepsilon.q)$$

for all $x, x' \in \mathbb{Z}_{2k}$ with $x' \neq x$. Then the symmetric three-voice successor correspondence S_H forbids non-trivial parallel q-motion between the middle and upper voices.

Proof. Let

$$\xi = \begin{bmatrix} c \\ b \\ a \end{bmatrix}, \quad \eta = \begin{bmatrix} c' \\ b' \\ a' \end{bmatrix}$$

be elements of X_H. Assume that

$$c - b = q, \quad c' - b' = q,$$

and that the middle voice moves non-trivially:

$$a' + b' \neq a + b.$$

Since $q \in X$, the middle–upper projection is active:

$$MU \in A(\xi).$$

Therefore the definition of S_H would require

$$\begin{aligned}(a' + b') + \varepsilon.q = \pi_{MU}(\eta) &\in S_X(\pi_{MU}(\xi)) \\ &= S_X((a + b) + \varepsilon.q).\end{aligned}$$

This contradicts the assumed two-voice prohibition of non-trivial parallel q-motion. Hence $\eta \notin S_H(\xi)$. □

Remark 2.8. The construction does not impose any additional rule on a distinguished interval. It simply applies the ordinary two-voice admitted-successor relation to every pair of voices which is consonant with respect to X. Consequently, every two-voice prohibition encoded by S_X is inherited by the corresponding active pair of voices in the three-voice system.

3. Three-Voice First-Species Counterpoint in the Fuxian Dichotomy

We specialize the preceding construction to the twelve-tone case $\mathbb{Z}_{12}$.

$$(K/D) = \big(\{0, 3, 4, 7, 8, 9\}/\{1, 2, 5, 6, 10, 11\}\big),$$

is the Fuxian interval dichotomy. It is a strong dichotomy, and its polarity is $p = T_2 \cdot 5, \quad p(x) = 2 + 5x$.

We retain the notation

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} = a + \varepsilon_1.b + \varepsilon_2.c \in \mathbb{Z}_{12}[\varepsilon_1, \varepsilon_2],$$

where b and c denote, respectively, the lower–middle and lower–upper intervals. The middle–upper interval is $c - b$.

Definition 3.1 (Okumura's Harmonic Mask)**.** Define

$$H = \left\{ \begin{bmatrix} c \\ b \end{bmatrix} \in K \times K : b - c \notin \{\pm 1, \pm 2\} \right\}.$$

In particular,

$$|H| = 28.$$

The harmonic mask for three voices in the Fuxian dichotomy is

$$H = H_{\text{full}} \sqcup H_{\text{inc}} \subset K \times K.$$

H_{full} is the complete sonorities, and H_{inc} is the incomplete sonorities.

$$H_{\text{full}} = \left\{ \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \begin{bmatrix} 8 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \end{bmatrix} \right\}$$

and

$$H_{\text{inc}} = \left\{ \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 9 \end{bmatrix} \right\}.$$

The associated set of three-voice Fuxian consonances is

$$K_H = \left\{ \begin{bmatrix} c \\ b \\ a \end{bmatrix} : a \in \mathbb{Z}_{12}, \ \begin{bmatrix} c \\ b \end{bmatrix} \in H \right\}.$$

Remark 3.2. Three-voice admitted successors in the Fuxian dichotomy are precisely the specialization of Section 2.1 to the case $2k = 12$. Hence it is unnecessary to restate the general formalism in full.

3.1. Admitted successors table in Fuxian dichotomy

For a source consonance

$$\xi = \begin{bmatrix} c \\ b \\ a \end{bmatrix} \in K_H$$

and a lower-voice displacement $j \in \mathbb{Z}_{12}$, a target consonance has the form

$$\eta = \begin{bmatrix} c' \\ b' \\ a + j \end{bmatrix}.$$

We write

$$F_{b,c}(j) = \left\{ (b', c') \in H : \begin{bmatrix} c \\ b \\ a \end{bmatrix} \not\longrightarrow \begin{bmatrix} c' \\ b' \\ a+j \end{bmatrix} \right\}.$$

Thus $F_{b,c}(j)$ is the set of forbidden target interval-pairs for the source pair (b, c) and lower-voice displacement j.

The computation below uses the symmetric Hichert-style maximized algorithm. Thus the ordinary Fuxian Hichert mechanism is applied to each active pairwise projection LM, LU, MU, where MU is active precisely when $c - b \in K$. The final successor set is obtained by maximizing the simultaneous three-voice intersection.

For compactness, we enumerate the elements of H_{Fux} as follows.

1	$(0,0)$	2	$(0,3)$	3	$(0,4)$	4	$(0,7)$
5	$(0,8)$	6	$(0,9)$	7	$(3,0)$	8	$(3,3)$
9	$(3,7)$	10	$(3,8)$	11	$(3,9)$	12	$(4,0)$
13	$(4,4)$	14	$(4,7)$	15	$(4,8)$	16	$(4,9)$
17	$(7,0)$	18	$(7,3)$	19	$(7,4)$	20	$(7,7)$
21	$(8,0)$	22	$(8,3)$	23	$(8,4)$	24	$(8,8)$
25	$(9,0)$	26	$(9,3)$	27	$(9,4)$	28	$(9,9)$

In the following table, an entry such as

$$\{4, 9, 20\}$$

denotes the forbidden target set

$$\{(0, 7), (3, 7), (7, 7)\}.$$

If

$$I = \{1, \dots, 28\},$$

then

$$\overline{A}$$

denotes $I \setminus A$. Hence

$$\overline{\{8, 9, 18\}}$$

means that every target is forbidden except the targets numbered $8, 9, 18$.

$j = 0$

source labels	forbidden target labels
$\{1,3,13,21\}$	$\overline{\{8,9,18,20,28\}}$
$\{2,6\}$	$\overline{\{9,10,12,13,14,15,20,21,23,24\}}$
$\{4\}$	$\overline{\{7,8,12,13,15,18,19,21,23,24\}}$
$\{5,12,23\}$	$\overline{\{8,11,26,28\}}$
$\{7,25\}$	$\overline{\{3,4,5,13,14,15,18,19,20,22,23,24\}}$
$\{8,28\}$	$\overline{\{3,4,5,13,14,15,20,23,24\}}$
$\{9\}$	$\overline{\{1,2,3,5,6,12,13,15,19,21,22,23,24\}}$
$\{10\}$	$\{5,6,7,8,9,10,11,15,16,24,25,26,27,28\}$
$\{11,26\}$	$\{2,6,7,8,9,10,11,16,18,22,25,26,27,28\}$
$\{14\}$	$\overline{\{7,8,17,18,25,27,28\}}$
$\{15,24\}$	$\overline{\{7,8,9,18,19,20,25,27,28\}}$
$\{16\}$	$\overline{\{7,9,10,17,19,20,25,27\}}$
$\{17\}$	$\overline{\{2,4,6,8,9,11,14,16,22,26,28\}}$
$\{18\}$	$\overline{\{1,3,4,9,13,14,15,21,24,25,27\}}$
$\{19\}$	$\overline{\{2,4,6,8,9,16,22,28\}}$
$\{20\}$	$\{4,6,7,9,10,13,14,16,17,18,19,20\}$
$\{22\}$	$\overline{\{1,3,5,7,9,12,13,14,15,19,20\}}$
$\{27\}$	$\overline{\{2,6,14,18,20\}}$

$j = 1$

source labels	forbidden target labels
$\{1,3,13,15,21,24\}$	$\{8,9,10,16,17,18,20,28\}$
$\{2,6\}$	$\{6,7,10,11,16,17,25,26,27,28\}$
$\{4\}$	$\{4,9,10,11,14,16,17,20,22,25,26,27,28\}$
$\{5,12,23\}$	$\{5,6,7,10,12,16,17,18,19,23\}$
$\{7,25\}$	$\{6,10,11,16,17,19,26,28\}$
$\{8,28\}$	$\{5,6,7,10,11,15,16,17,19,24,25,26,27,28\}$
$\{9\}$	$\overline{\{2,6,7,8,18,19,22,25,27,28\}}$
$\{10\}$	$\{5,6,10,11,15,16,24,28\}$
$\{11,16,26\}$	$\varnothing$
$\{14\}$	$\{2,4,6,7,9,11,14,16,17,20,25,26\}$
$\{17\}$	$\{17,18,19,20\}$
$\{18\}$	$\{5,6,7,10,12,16,17,18,19,20,23\}$
$\{19\}$	$\{2,6,7,11,16,17,18,19,20,25,26\}$
$\{20\}$	$\overline{\{2,3,5,12,13,15,21,22,23,24,25,27\}}$
$\{22\}$	$\{4,10,11,16,17,21,22,23,24,25,26,27,28\}$
$\{27\}$	$\{4,10,11,16,17,22,26,27\}$

$j = 2$

source labels	forbidden target labels
$\{1,3,13,21\}$	$\overline{\{8,9,18,20,28\}}$
$\{2,6\}$	$\{4,9,10,11,14,19,20,25,26,27,28\}$
$\{4\}$	$\{4,9,10,11,14,16,17,20,22,25,26,27,28\}$
$\{5,12,23\}$	$\overline{\{8,11,26,28\}}$
$\{7,25\}$	$\{6,10,11,14,16,19,26,28\}$
$\{8,28\}$	$\{3,4,6,7,9,10,13,14,16,17,18,19,20,23\}$
$\{9\}$	$\{4,8,9,10,14,16,17,18,20,28\}$
$\{10\}$	$\{5,6,10,11,15,16,24,28\}$
$\{11,26\}$	$\varnothing$
$\{14\}$	$\overline{\{7,8,18,19,25,27,28\}}$
$\{15,24\}$	$\overline{\{7,8,9,18,19,20,25,27,28\}}$
$\{16\}$	$\overline{\{7,8,9,10,11,17,18,19,20,25,26,27,28\}}$
$\{17\}$	$\overline{\{2,4,6,8,9,11,14,16,22,26,28\}}$
$\{18\}$	$\{5,6,7,10,12,16,17,18,19,20,23\}$
$\{19\}$	$\overline{\{2,4,6,8,9,14,22,28\}}$
$\{20\}$	$\{4,6,7,9,10,14,16,17,18,19,20,24\}$
$\{22\}$	$\{4,10,11,16,17,21,22,23,24,25,26,27,28\}$
$\{27\}$	$\overline{\{2,6,8,9,14,18,20,28\}}$

$j = 3$

source labels	forbidden target labels
$\{1,3,13,15,21,24\}$	$\{8,9,10,16,17,18,20,28\}$
$\{2,6\}$	$\overline{\{9,10,12,13,14,15,20,21,23,24\}}$
$\{4\}$	$\overline{\{7,8,12,13,15,18,19,21,23,24\}}$
$\{5,12,23\}$	$\{5,6,7,10,12,16,17,18,19,23\}$
$\{7,25\}$	$\overline{\{3,4,5,13,14,15,18,19,20,22,23,24\}}$
$\{8,28\}$	$\overline{\{3,4,5,13,14,15,20,23,24\}}$
$\{9\}$	$\overline{\{2,6,18,19,22\}}$
$\{10\}$	$\{5,6,7,8,9,10,11,15,16,24,25,26,27,28\}$
$\{11,26\}$	$\{2,6,7,8,9,10,11,16,18,22,25,26,27,28\}$
$\{14\}$	$\{4,9,10,11,14,19,20,26\}$
$\{16\}$	$\{2,6,8,11,16,18,22,26,28\}$
$\{17\}$	$\{17,18,19,20\}$
$\{18\}$	$\overline{\{1,3,4,9,13,14,15,21,24,25,27\}}$
$\{19\}$	$\{10,11,14,17,18,19,20,26\}$
$\{20\}$	$\overline{\{4,6,7,9,10,13,14,16,17,18,19,20\}}$
$\{22\}$	$\overline{\{1,3,5,7,9,12,13,14,15,19,20\}}$
$\{27\}$	$\{4,7,8,9,10,11,16,17,22,25,26,27,28\}$

$j = 6$

source labels	forbidden target labels
$\{1,3,13,21\}$	$\overline{\{8,9,18,20,28\}}$
$\{2,6\}$	$\overline{\{9,10,12,13,14,15,20,21,23,24\}}$
$\{4\}$	$\overline{\{7,8,12,13,15,18,19,21,23,24\}}$
$\{5,12,23\}$	$\overline{\{8,11,26,28\}}$
$\{7,25\}$	$\overline{\{3,4,5,13,14,15,18,19,20,22,23,24\}}$
$\{8,28\}$	$\overline{\{3,4,5,13,14,15,20,23,24\}}$
$\{9\}$	$\overline{\{1,2,3,5,6,12,13,15,19,21,22,23,24\}}$
$\{10\}$	$\{5,6,7,8,9,10,11,15,16,24,25,26,27,28\}$
$\{11,26\}$	$\{2,6,7,8,9,10,11,16,18,22,25,26,27,28\}$
$\{14\}$	$\overline{\{7,8,17,18,25,27,28\}}$
$\{15,24\}$	$\overline{\{7,8,9,18,19,20,25,27,28\}}$
$\{16\}$	$\overline{\{7,9,10,17,19,20,25,27\}}$
$\{17\}$	$\overline{\{2,4,6,8,9,11,14,16,22,26,28\}}$
$\{18\}$	$\overline{\{1,3,4,9,13,14,15,21,24,25,27\}}$
$\{19\}$	$\overline{\{2,4,6,8,9,16,22,28\}}$
$\{20\}$	$\{4,6,7,9,10,13,14,16,17,18,19,20\}$
$\{22\}$	$\overline{\{1,3,5,7,9,12,13,14,15,19,20\}}$
$\{27\}$	$\overline{\{2,6,14,18,20\}}$

$j = 4$

source labels	forbidden target labels
$\{1,3,13,21\}$	$\overline{\{8,9,18,20,28\}}$
$\{2,6\}$	$\{6,7,10,11,16,17,25,26,27,28\}$
$\{4\}$	$\{4,9,10,11,14,16,17,20,22,25,26,27,28\}$
$\{5,12,23\}$	$\overline{\{8,11,26,28\}}$
$\{7,25\}$	$\{6,10,11,16,17,19,26,28\}$
$\{8,28\}$	$\{5,6,7,10,11,15,16,17,19,24,25,26,27,28\}$
$\{9\}$	$\{4,8,9,10,14,16,17,18,20,28\}$
$\{10\}$	$\{5,6,10,11,15,16,24,28\}$
$\{11,26\}$	$\varnothing$
$\{14\}$	$\overline{\{8,10,18,19,27,28\}}$
$\{15,24\}$	$\overline{\{7,8,9,18,19,20,25,27,28\}}$
$\{16\}$	$\overline{\{7,8,9,10,11,17,18,19,20,25,26,27,28\}}$
$\{17\}$	$\overline{\{2,4,6,8,9,11,14,16,22,26,28\}}$
$\{18\}$	$\{5,6,7,10,12,16,17,18,19,20,23\}$
$\{19\}$	$\overline{\{4,8,9,14,22,28\}}$
$\{20\}$	$\overline{\{2,3,5,12,13,15,21,22,23,24,25,27\}}$
$\{22\}$	$\{4,10,11,16,17,21,22,23,24,25,26,27,28\}$
$\{27\}$	$\overline{\{2,6,8,9,14,18,20,28\}}$

$j = 7$

source labels	forbidden target labels
$\{1,3,13,15,21,24\}$	$\{8,9,10,16,17,18,20,28\}$
$\{2,6\}$	$\{6,7,10,11,16,17,25,26,27,28\}$
$\{4\}$	$\{4,9,10,11,14,16,17,20,22,25,26,27,28\}$
$\{5,12,23\}$	$\{5,6,7,10,12,16,17,18,19,23\}$
$\{7,25\}$	$\{6,10,11,16,17,19,26,28\}$
$\{8,28\}$	$\{5,6,7,10,11,15,16,17,19,24,25,26,27,28\}$
$\{9\}$	$\overline{\{2,6,7,8,18,19,22,25,27,28\}}$
$\{10\}$	$\{5,6,10,11,15,16,24,28\}$
$\{11,16,26\}$	$\varnothing$
$\{14\}$	$\{2,4,6,7,9,11,14,16,17,20,25,26\}$
$\{17\}$	$\{17,18,19,20\}$
$\{18\}$	$\{5,6,7,10,12,16,17,18,19,20,23\}$
$\{19\}$	$\{2,6,7,11,16,17,18,19,20,25,26\}$
$\{20\}$	$\overline{\{2,3,5,12,13,15,21,22,23,24,25,27\}}$
$\{22\}$	$\{4,10,11,16,17,21,22,23,24,25,26,27,28\}$
$\{27\}$	$\{4,10,11,16,17,22,26,27\}$

$j = 8$

source labels	forbidden target labels
$\{1,3,13,21\}$	$\overline{\{8,9,18,20,28\}}$
$\{2,6\}$	$\{4,9,10,11,14,19,20,25,26,27,28\}$
$\{4\}$	$\{4,9,10,11,14,16,17,20,22,25,26,27,28\}$
$\{5,12,23\}$	$\overline{\{8,11,26,28\}}$
$\{7,25\}$	$\{6,10,11,14,16,19,26,28\}$
$\{8,28\}$	$\{3,4,6,7,9,10,13,14,16,17,18,19,20,23\}$
$\{9\}$	$\{4,8,9,10,14,16,17,18,20,28\}$
$\{10\}$	$\{5,6,10,11,15,16,24,28\}$
$\{11,26\}$	$\varnothing$
$\{14\}$	$\overline{\{7,8,18,19,25,27,28\}}$
$\{15,24\}$	$\overline{\{7,8,9,18,19,20,25,27,28\}}$
$\{16\}$	$\overline{\{7,8,9,10,11,17,18,19,20,25,26,27,28\}}$
$\{17\}$	$\overline{\{2,4,6,8,9,11,14,16,22,26,28\}}$
$\{18\}$	$\{5,6,7,10,12,16,17,18,19,20,23\}$
$\{19\}$	$\overline{\{2,4,6,8,9,14,22,28\}}$
$\{20\}$	$\{4,6,7,9,10,14,16,17,18,19,20,24\}$
$\{22\}$	$\{4,10,11,16,17,21,22,23,24,25,26,27,28\}$
$\{27\}$	$\overline{\{2,6,8,9,14,18,20,28\}}$

$j = 5$

source labels	forbidden target labels
$\{1,3,13,15,21,24\}$	$\{8,9,10,16,17,18,20,28\}$
$\{2,6\}$	$\{4,9,10,11,14,19,20,25,26,27,28\}$
$\{4\}$	$\{4,9,10,11,14,16,17,20,22,25,26,27,28\}$
$\{5,12,23\}$	$\{5,6,7,10,12,16,17,18,19,23\}$
$\{7,25\}$	$\{6,10,11,14,16,19,26,28\}$
$\{8,28\}$	$\{3,4,6,7,9,10,13,14,16,17,18,19,20,23\}$
$\{9\}$	$\overline{\{2,6,7,8,18,19,22,25,27,28\}}$
$\{10\}$	$\{5,6,10,11,15,16,24,28\}$
$\{11,16,26\}$	$\varnothing$
$\{14\}$	$\{4,9,10,11,14,16,17,20,26\}$
$\{17\}$	$\{17,18,19,20\}$
$\{18\}$	$\{5,6,7,10,12,16,17,18,19,20,23\}$
$\{19\}$	$\{10,11,16,17,18,19,20,26\}$
$\{20\}$	$\{4,6,7,9,10,14,16,17,18,19,20,24\}$
$\{22\}$	$\{4,10,11,16,17,21,22,23,24,25,26,27,28\}$
$\{27\}$	$\{4,10,11,16,17,22,26,27\}$

$j = 9$

source labels	forbidden target labels
$\{1, 3, 13, 15, 21, 24\}$	$\{8, 9, 10, 16, 17, 18, 20, 28\}$
$\{2, 6\}$	$\{9, 10, 12, 13, 14, 15, 20, 21, 23, 24\}$
$\{4\}$	$\{7, 8, 12, 13, 15, 18, 19, 21, 23, 24\}$
$\{5, 12, 23\}$	$\{5, 6, 7, 10, 12, 16, 17, 18, 19, 23\}$
$\{7, 25\}$	$\{3, 4, 5, 13, 14, 15, 18, 19, 20, 22, 23, 24\}$
$\{8, 28\}$	$\{3, 4, 5, 13, 14, 15, 20, 23, 24\}$
$\{9\}$	$\{2, 6, 18, 19, 22\}$
$\{10\}$	$\{5, 6, 7, 8, 9, 10, 11, 15, 16, 24, 25, 26, 27, 28\}$
$\{11, 26\}$	$\{2, 6, 7, 8, 9, 10, 11, 16, 18, 22, 25, 26, 27, 28\}$
$\{14\}$	$\{4, 9, 10, 11, 14, 19, 20, 26\}$
$\{16\}$	$\{2, 6, 8, 11, 16, 18, 22, 26, 28\}$
$\{17\}$	$\{17, 18, 19, 20\}$
$\{18\}$	$\{1, 3, 4, 9, 13, 14, 15, 21, 24, 25, 27\}$
$\{19\}$	$\{10, 11, 14, 17, 18, 19, 20, 26\}$
$\{20\}$	$\{4, 6, 7, 9, 10, 13, 14, 16, 17, 18, 19, 20\}$
$\{22\}$	$\{1, 3, 5, 7, 9, 12, 13, 14, 15, 19, 20\}$
$\{27\}$	$\{4, 7, 8, 9, 10, 11, 16, 17, 22, 25, 26, 27, 28\}$

$j = 10$

source labels	forbidden target labels
$\{1, 3, 13, 21\}$	$\{8, 9, 18, 20, 28\}$
$\{2, 6\}$	$\{6, 7, 10, 11, 16, 17, 25, 26, 27, 28\}$
$\{4\}$	$\{4, 9, 10, 11, 14, 16, 17, 20, 22, 25, 26, 27, 28\}$
$\{5, 12, 23\}$	$\{8, 11, 26, 28\}$
$\{7, 25\}$	$\{6, 10, 11, 16, 17, 19, 26, 28\}$
$\{8, 28\}$	$\{5, 6, 7, 10, 11, 15, 16, 17, 19, 24, 25, 26, 27, 28\}$
$\{9\}$	$\{4, 8, 9, 10, 14, 16, 17, 18, 20, 28\}$
$\{10\}$	$\{5, 6, 10, 11, 15, 16, 24, 28\}$
$\{11, 26\}$	$\varnothing$
$\{14\}$	$\{8, 10, 18, 19, 27, 28\}$
$\{15, 24\}$	$\{7, 8, 9, 18, 19, 20, 25, 27, 28\}$
$\{16\}$	$\{7, 8, 9, 10, 11, 17, 18, 19, 20, 25, 26, 27, 28\}$
$\{17\}$	$\{2, 4, 6, 8, 9, 11, 14, 16, 22, 26, 28\}$
$\{18\}$	$\{5, 6, 7, 10, 12, 16, 17, 18, 19, 20, 23\}$
$\{19\}$	$\{4, 8, 9, 14, 22, 28\}$
$\{20\}$	$\{2, 3, 5, 12, 13, 15, 21, 22, 23, 24, 25, 27\}$
$\{22\}$	$\{4, 10, 11, 16, 17, 21, 22, 23, 24, 25, 26, 27, 28\}$
$\{27\}$	$\{2, 6, 8, 9, 14, 18, 20, 28\}$

$j = 11$

source labels	forbidden target labels
$\{1, 3, 13, 15, 21, 24\}$	$\{8, 9, 10, 16, 17, 18, 20, 28\}$
$\{2, 6\}$	$\{4, 9, 10, 11, 14, 19, 20, 25, 26, 27, 28\}$
$\{4\}$	$\{4, 9, 10, 11, 14, 16, 17, 20, 22, 25, 26, 27, 28\}$
$\{5, 12, 23\}$	$\{5, 6, 7, 10, 12, 16, 17, 18, 19, 23\}$
$\{7, 25\}$	$\{6, 10, 11, 14, 16, 19, 26, 28\}$
$\{8, 28\}$	$\{3, 4, 6, 7, 9, 10, 13, 14, 16, 17, 18, 19, 20, 23\}$
$\{9\}$	$\{2, 6, 7, 8, 18, 19, 22, 25, 27, 28\}$
$\{10\}$	$\{5, 6, 10, 11, 15, 16, 24, 28\}$
$\{11, 16, 26\}$	$\varnothing$
$\{14\}$	$\{4, 9, 10, 11, 14, 16, 17, 20, 26\}$
$\{17\}$	$\{17, 18, 19, 20\}$
$\{18\}$	$\{5, 6, 7, 10, 12, 16, 17, 18, 19, 20, 23\}$
$\{19\}$	$\{10, 11, 16, 17, 18, 19, 20, 26\}$
$\{20\}$	$\{4, 6, 7, 9, 10, 14, 16, 17, 18, 19, 20, 24\}$
$\{22\}$	$\{4, 10, 11, 16, 17, 21, 22, 23, 24, 25, 26, 27, 28\}$
$\{27\}$	$\{4, 10, 11, 16, 17, 22, 26, 27\}$

The rows are grouped by identical forbidden-transition profiles. For example, the first row applies simultaneously to the six source pairs numbered

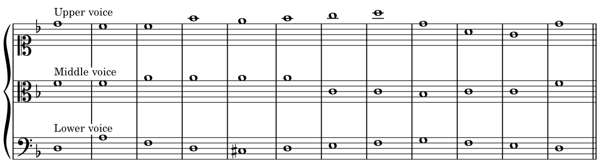
$$1, 3, 12, 13, 21, 23,$$

namely

$$(0, 0), (0, 4), (4, 0), (4, 4), (8, 0), (8, 4).$$

Example 3.3. The example [1] of three-voice first-species counterpoint is

$$\xi_1 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \xi_2 = \begin{bmatrix} 3 \\ 8 \\ 9 \end{bmatrix}, \xi_3 = \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix}, \xi_4 = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix},$$
$$\xi_5 = \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}, \xi_6 = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}, \xi_7 = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}, \xi_8 = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix},$$
$$\xi_9 = \begin{bmatrix} 7 \\ 3 \\ 7 \end{bmatrix}, \xi_{10} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}, \xi_{11} = \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix}, \xi_{12} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$



All 11 successors are admissible.

4. Toward a United Counterpoint–Harmony Theory (UCHT)

The three-voice Fuxian model suggests a natural way to unify local harmony and first-species counterpoint. The central idea is to regard a three-voice sonority not merely as a chord, and not merely as a contrapuntal intervallic configuration, but as a single object carrying both vertical and horizontal information. In this framework, harmonic rules are interpreted as conditions on individual sonorities. They determine which vertical configurations are admissible. For instance, the prohibition of the fourth above the bass is not added as an external rule; it follows from the requirement that each upper voice form a Fuxian consonance with the lower voice. At the same time, first-inversion sonorities remain possible, since the interval between the two upper voices is not forced to obey the same bass-oriented restriction. Contrapuntal rules are interpreted as conditions on transitions between admissible sonorities. The successor relation is obtained by applying the two-voice Fuxian admitted-successor mechanism to each relevant pair of voices. In this way, the avoidance of parallel fifths between any two voices is not imposed separately. It is inherited from the ordinary two-

[1] Although this is my own construction, the Lower-voice is quoted from the subject of J. S. Bach's "The Art of Fugue", BWV 1080.

voice Fuxian theory whenever the corresponding pair of voices forms a consonant interval. Thus the same structure contains two types of information. Its vertices encode harmonic admissibility, while its directed edges encode contrapuntal admissibility. Harmony is therefore described by the allowed vertical objects, and counterpoint by the allowed motions between them. This also suggests a precise relation between the two theories. Counterpoint may be regarded as the more refined structure, since it remembers the individual voices, their positions, and their directed motions. Harmony may be regarded as a quotient or abstraction of this structure, obtained by forgetting some of the voice-specific information and retaining only chordal type or harmonic function. Consequently, a literal one-to-one correspondence between all traditional harmonic rules and all traditional contrapuntal rules would be too strong. Traditional harmony often identifies several distinct voiced configurations as the same chord, whereas counterpoint distinguishes them. A more appropriate formulation is that voiced harmonic rules correspond to admissibility conditions on vertical configurations, while contrapuntal rules correspond to admissibility conditions on directed transitions. The resulting theory can be summarized as follows: harmony is the vertical shadow of counterpoint, and counterpoint is the horizontal refinement of harmony. The three-voice Fuxian model provides a finite algebraic setting in which this relationship can be made explicit.

Harmonic statement	Counterpoint-theoretic formulation
Allowed chord	$(b, c) \in H$
Bass fourth forbidden	$b, c \neq 5$, since $H \subset K \times K$
Incomplete chord allowed	$b = 0$, $c = 0$, or $b = c$
First inversion allowed	$c - b \in \{5, 6\}$ may occur
Parallel fifth forbidden	$7 \to 7$ is forbidden in every active pairwise projection
Voice-leading admissibility	$\eta \in S_H(\xi)$

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